

SMALL MAXIMAL SPACES OF NON-INVERTIBLE MATRICES

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ABSTRACT

The *rank* of a vector space \mathcal{A} of $n \times n$ -matrices is by definition the maximal rank of an element of \mathcal{A} . The space \mathcal{A} is called *rank-critical* if any matrix space that properly contains \mathcal{A} has a strictly higher rank. This paper exhibits a sufficient condition for rank-criticality, which is then used to prove that the images of certain Lie algebra representations are rank-critical. A rather counter-intuitive consequence, and the main novelty in this paper, is that for infinitely many n , there exists an eight-dimensional rank-critical space of $n \times n$ -matrices having generic rank $n - 1$, or, in other words: an eight-dimensional maximal space of non-invertible matrices. This settles the question, posed by Fillmore, Laurie, and Radjavi in 1985, of whether such a maximal space can have dimension smaller than n . Another consequence is that the image of the adjoint representation of any semisimple Lie algebra is rank-critical; in both results, the ground field is assumed to have characteristic zero.

1. Results

This paper deals with linear subspaces of $\text{End}(V)$, the space of K -linear maps from an n -dimensional vector space V over a field K into itself. The (*generic*) *rank* of such a subspace \mathcal{A} , denoted $\text{rk } \mathcal{A}$, is by definition the highest rank of an element of \mathcal{A} , and we call \mathcal{A} *rank-critical* if any linear subspace \mathcal{B} of $\text{End}(V)$ that properly contains \mathcal{A} has $\text{rk } \mathcal{B} > \text{rk } \mathcal{A}$. Note that a space \mathcal{A} is maximal among the *singular spaces* – that is, those that only contain non-invertible matrices – if and only if \mathcal{A} is rank-critical of rank $n - 1$; in this case we call \mathcal{A} a *maximal singular space*. The main results of this paper are the following.

THEOREM 1. *Let K be a field of characteristic zero, let $m \geq 3$ be an integer, and let e be a positive integer. Then the image of the representation of $\mathfrak{sl}_m(K)$ on the space $V = K[x_1, \dots, x_m]_{em}$ of homogeneous polynomials of degree em is a maximal singular subspace of $\text{End}(V)$.*

In particular, taking $m = 3$, we find that for every n of the form $\binom{3e+2}{2}$, $e \geq 1$, there exists an eight-dimensional maximal singular space of $n \times n$ -matrices.

THEOREM 2. *For any semisimple Lie algebra \mathfrak{g} over a field of characteristic zero, $\text{ad } \mathfrak{g}$ is a rank-critical subspace of $\text{End}(\mathfrak{g})$.*

Theorems 1 and 2 are consequences of the following proposition, in whose formulation – as in the rest of this paper – we use the following standard terminology: if a group G acts on a set X , then a subset Y of X is called *stable under the action*

of G (or simply G -stable) if $gy \in Y$ for all $y \in Y$ and $g \in G$. If not obvious from the context, the relevant group action will be made explicit.

PROPOSITION 3. *Let \mathcal{A} be a subspace of $\text{End}(V)$, and suppose that the cardinality $|K|$ of K is strictly larger than $\text{rk } \mathcal{A} =: r$. Set $\mathcal{A}_{\text{reg}} := \{A \in \mathcal{A} \mid \text{rk } A = r\}$ and define the space*

$$\text{RND}(\mathcal{A}) := \{B \in \text{End}(V) \mid B \ker A \subseteq \text{im } A \text{ for all } A \in \mathcal{A}_{\text{reg}}\}$$

of rank-neutral directions of \mathcal{A} . Then $\text{RND}(\mathcal{A}) \supseteq \mathcal{A}$; if equality holds, then \mathcal{A} is rank-critical. If, moreover, a group G acts linearly on V and if \mathcal{A} is stable under the conjugation action of G on $\text{End}(V)$, then $\text{RND}(\mathcal{A})$ is also stable under the conjugation action of G .

The proof of Proposition 3 given in Section 3 is based on an elementary but useful sufficient condition for maximality of vector spaces in an arbitrary affine variety embedded in a vector space. In Section 4, we apply Proposition 3 to images of Lie algebra representations; Theorems 1 and 2 are proved there. Section 5 lists some computer results on rank-criticality of representations of semisimple Lie algebras; in particular, Theorem 2 arose as a conjecture from these experiments.

2. Introduction and motivation

The direct motivation for this paper is the question, posed by Fillmore *et al.* in 1985 [8], of whether a maximal singular subspace of $\text{End}(V)$ can have dimension smaller than n . We briefly review three well-known constructions of maximal singular spaces that led them to raise this question.

EXAMPLE 4. Fix subspaces W, U of V , of dimensions $k - 1$ and k , respectively, and set

$$\mathcal{A} := \{A \in \text{End}(V) \mid AU \subseteq W\}.$$

Then \mathcal{A} is a singular space of dimension

$$k(k - 1) + (n - k)n = n^2 - kn + k^2 - k.$$

Moreover, it not hard to see that \mathcal{A} is maximal. We follow Eisenbud and Harris [7] in calling \mathcal{A} and all its subspaces *compression spaces*, as they ‘compress’ U into W .

EXAMPLE 5. Suppose that n is odd, take $V = K^n$, and let \mathcal{A} be the space of all skew-symmetric matrices. As any skew-symmetric matrix has even rank, the space \mathcal{A} is singular, and it was observed in [8] that \mathcal{A} is maximal for all odd $n \geq 3$, under the assumption that $|K| \geq 3$. It is easy to see that \mathcal{A} is not a compression space.

In both the examples above, the dimension of \mathcal{A} is quadratic in n . An ingenious construction of smaller maximal singular spaces is the following, attributed to Bob Paré in [8], and also appearing in [18].

EXAMPLE 6. Take $V = K^n$ and fix n skew-symmetric $n \times n$ -matrices A_1, \dots, A_n . Let ϕ be the linear map from K^n into the space $M_n(K)$ of $n \times n$ -matrices over K sending x to the matrix with columns A_1x, A_2x, \dots, A_nx . Then $\phi(K^n)$ is a singular

space in $M_n(K)$ because $x^t\phi(x) = 0$ for all $x \in K^n$. Moreover, if we denote by $E_{i,j}$ the matrix with zeroes everywhere except for a 1 in position (i, j) , then in the particular case where $|K| \geq 3$, $A_i = E_{i,i+1} - E_{i+1,i}$ for $i < n$, and $A_n = E_{n,1} - E_{1,n}$, Fillmore *et al.* have shown that \mathcal{A} is maximal [8].

Many results in the literature exhibit sufficient conditions for a singular space \mathcal{A} to be a compression space: Dieudonné [5] showed that every singular space of dimension at least $n^2 - n$ either has a non-trivial common kernel or is dual to a space with a common kernel. Under the assumption that $|K|$ is at least $2n - 2$, this result is sharpened as follows in [8]: if the dimension of \mathcal{A} is greater than $n^2 - 2n + 2$ (which is the dimension of a compression space with $k = 2$), then \mathcal{A} or its dual has a common kernel. A condition of a different kind is that \mathcal{A} be spanned by rank-one matrices; then a combinatorial argument shows that \mathcal{A} is a compression space [13, 18]. Analogues of these questions for (skew-)symmetric matrices and for rank-critical spaces have also been studied extensively in the literature [1, 9, 13, 15–17, 20].

Yet another result of this type is part of the Kronecker–Weierstrass theory of matrix pencils [11] for $K = \mathbb{C}$. It is not hard, using the Kronecker–Weierstrass approach, to generalise their result as follows: if $|K| \geq n$, then any two-dimensional singular space is a compression space. Another proof of this fact is given in [7]. This statement shows that if $|K| \geq n > 2$, then a maximal singular space cannot have dimension 2. On the other hand, Example 6 shows that there do exist maximal singular spaces in $\text{End}(V)$ of dimension $\dim(V)$, and this led Fillmore *et al.* to put forward their question above.

We conclude this section with a note on the ground field K . In most of the results quoted above, there is a condition on the cardinality of K . The following example shows that this condition is necessary in the statement about two-dimensional singular spaces.

EXAMPLE 7. Suppose that K is a finite field with q elements labelled c_1, \dots, c_q . Then the two-dimensional subspace \mathcal{A} of $M_{q+1}(K)$ spanned by the diagonal matrices $A = \text{diag}(1, 1, \dots, 1, 0)$ and $B = \text{diag}(c_1, c_2, \dots, c_q, 1)$ is singular: A itself is singular and for any $i \in \{1, \dots, q\}$ the i th diagonal entry of the matrix $B - c_i A$ is zero. However, if L/K is any proper field extension and $c \in L \setminus K$, then $B - cA$, viewed as an L -linear map on $L \otimes_K V$, is invertible. It follows that $L \otimes_K \mathcal{A}$ is not a singular space.

In this paper we will be interested only in matrix spaces whose generic ranks do not change upon field extension. This can be ensured as follows: suppose that \mathcal{A} is a matrix space of generic rank r , spanned by matrices A_1, \dots, A_d . Then any $(r+1) \times (r+1)$ -minor of the linear combination $t_1 A_1 + \dots + t_d A_d$ is a homogeneous polynomial of degree $r+1$ in the t_i that vanishes for all $(t_1, \dots, t_d) \in K^d$. But it is easy to see that a non-zero homogeneous polynomial in $K[t_1, \dots, t_d]$ of degree $r+1$ can vanish entirely on K^d only if $|K| \leq r$. Hence if $|K| > r$, then any $(r+1) \times (r+1)$ -minor of $t_1 A_1 + \dots + t_k A_k$ is the zero polynomial. As the common zeroes of the $(r+1) \times (r+1)$ -minors form the variety of matrices of rank at most r – indeed, the ideal of polynomials vanishing on this variety is generated by these minors (see [4]) but we do not need that here – we find that $L \otimes_K \mathcal{A}$ has generic rank r for every field extension L/K . If, moreover, we can show that $L \otimes_K \mathcal{A}$ is rank-critical, then

a *fortiori* \mathcal{A} is rank-critical. Hence, for the type of matrix spaces that we will study here, it is no severe restriction to assume that K is, in fact, algebraically closed. This somewhat simplifies the discussion in the following section.

3. Maximality of vector spaces in affine varieties

Let K be an algebraically closed field, and let M be a vector space over K . Let Z be an affine algebraic variety in M , and let N be a vector subspace of M contained in Z . We want a sufficient condition for N to be maximal among the subspaces of M contained in Z . Therefore, we set

$$U := \{m \in M \mid N + Km \subseteq Z\};$$

then U , too, is an affine variety in M . Maximality of N among the subspaces of M contained in Z is equivalent to $U = N$. In principle, this equality can be tested using Gröbner basis techniques, but U may be hard to compute, even for moderately complicated Z . We therefore set out to find *linear* sufficient conditions, as follows. Let Z_{reg} be the set of smooth points of Z , and set $N_{\text{reg}} := N \cap Z_{\text{reg}}$. We make the following assumption:

$$N_{\text{reg}} \neq \emptyset. \quad (*)$$

Now let

$$T_{N_{\text{reg}}}Z := \bigcap_{n \in N_{\text{reg}}} T_nZ$$

be the intersection of all tangent spaces to Z at points of N_{reg} , where each T_nZ is viewed as a vector subspace (through the origin) of M . We can now state the sufficient condition for maximality of N among the subspaces of M contained in Z that was promised in Section 1. Using the last part of the following proposition, the verification of this sufficient condition may be considerably simplified if a group acts linearly on M , Z and N .

PROPOSITION 8. (i) *The K -vector space $T_{N_{\text{reg}}}Z$ contains U .*

(ii) *In particular, if $T_{N_{\text{reg}}}Z = N$, then N is maximal among the subspaces of M contained in Z .*

(iii) *If a group G acts linearly on M and if Z and N are both G -stable, then so is $T_{N_{\text{reg}}}Z$.*

Proof. For $m \in U$ and $n \in N_{\text{reg}}$ the line $\{n + tm \mid t \in L\}$ lies in Z , and a *fortiori* m is tangent to Z at n . We conclude that $m \in T_{N_{\text{reg}}}Z$, as claimed in (i). Statement (ii) is now immediate. For (iii), let $m \in T_{N_{\text{reg}}}Z$, $g \in G$, and $n \in N_{\text{reg}}$. Then $g^{-1}n \in N \cap Z_{\text{reg}} = N_{\text{reg}}$, and g maps the tangent space $T_{g^{-1}n}Z$ isomorphically onto T_nZ . As m lies in the former by assumption, gm lies in the latter. This shows that $gm \in T_{N_{\text{reg}}}Z$. \square

We will apply Proposition 8 to the setting of Section 1: $M = \text{End}(V)$, where V is a vector space over K , $N = \mathcal{A}$ is a subspace of $\text{End}(V)$ of generic rank r , and $Z = R_r$ is the variety of linear maps $V \rightarrow V$ of rank at most r . It is well known that the smooth points of R_r are precisely the maps of rank exactly r [14, Example 14.16], and we recall the following characterisation of the tangent spaces to R_r at smooth points, which can also be found in [14].

LEMMA 9. For $A \in R_r$ of rank r , the tangent space $T_A R_r$ is equal to $\{B \in \text{End}(V) \mid B \ker A \subseteq \text{im } A\}$.

Proof. We give only an intuitive argument: for B to lie in $T_A R_r$, it is necessary and sufficient that there be, for every $v \in \ker(A)$, a vector $w \in V$ for which $(A + \varepsilon B)(v + \varepsilon w) = 0$ modulo ε^2 . The coefficient of ε in this expression is $Aw + Bv$, so that the existence of such a w is equivalent to $Bv \in \text{im}(A)$. \square

From now on, we write $\text{RND}(\mathcal{A})$ for $T_{\mathcal{A}_{\text{reg}}} R_r$; Lemma 9 shows that this definition agrees with the definition of the rank-neutral directions in Proposition 3. Proposition 3 for algebraically closed K is now a direct consequence of Proposition 8 and Lemma 9. For K not algebraically closed, it is not hard to see, using the note on the ground field at the end of the Introduction, that the conditions of Proposition 3 imply the same conditions with K replaced by its algebraic closure L . Hence we find that $L \otimes_K \mathcal{A}$ is rank-critical, and therefore so is \mathcal{A} .

EXAMPLE 10. Proposition 3 provides another proof (in the case where $|K| \geq \dim V$) of rank-criticality of compression spaces. Indeed, in the notation of Example 4, suppose that $B \in \text{End}(V)$ does not map U into W , and let $u \in U$ be such that $Bu \notin W$. It is then not hard to construct an $A \in \mathcal{A}$ with $\ker A = Ku$ and $\text{im } A \not\supseteq Bu$; indeed, choosing subspaces U', W', R, S of V such that $V = U \oplus U' = W \oplus W'$ and $U = Ku \oplus R$ and $W' = K(Bu) \oplus S$, we can take for A any linear map with $Au = 0$ that maps R isomorphically onto W and U' isomorphically onto S . We find that $\text{RND}(\mathcal{A}) = \mathcal{A}$ and \mathcal{A} is a maximal singular space, by Proposition 3.

4. A construction of rank-critical spaces

The singular space of Example 5 is closed under the commutator, and so are the compression spaces of Example 4 if $W \subseteq U$. This suggests the study of the following situation: let K be an algebraically closed field, let G be an affine algebraic group over K , and let $\rho : G \rightarrow \text{GL}(V)$ be a finite-dimensional rational representation. (For notions concerning algebraic groups, their Lie-algebras, and their representations, we refer to [2, 22].) Let \mathfrak{g} be the Lie algebra of G , and denote the corresponding representation $\mathfrak{g} \rightarrow \text{End}(V)$ also by ρ . Set $r := \text{rk } \rho(\mathfrak{g})$. Now G acts on $\text{End}(V)$ by $gA := \rho(g)A\rho(g)^{-1}$, and both $\rho(\mathfrak{g})$ and the variety R_r of linear maps of rank at most r are G -stable. Proposition 3 implies that: *the rank-neutral directions of $\rho(\mathfrak{g})$ form a G -module, and if $\text{RND}(\rho(\mathfrak{g})) = \rho(\mathfrak{g})$, then $\rho(\mathfrak{g})$ is rank-critical.* In the rest of this section we assume that $\text{char } K = 0$, so that we can use the well-known representation theory of semisimple Lie algebras to construct rank-critical spaces.

EXAMPLE 11. Recall Example 5. Here G is the group O_n of orthogonal matrices, $\mathfrak{g} = \mathfrak{o}_n$, ρ is the identity, and $V = K^n$ is the standard \mathfrak{o}_n -module. It is well known that $\text{End}(V)$ is the direct sum of three irreducible O_n -modules: the space \mathfrak{o}_n of skew-symmetric matrices, the scalar multiples of the identity I , and the space of symmetric matrices with trace 0. We now show that the last two modules are not contained in $\text{RND}(\mathfrak{o}_n)$. Choose

$$Y := \text{diag}(1, -1, 0, \dots, 0) \quad \text{and} \quad X := \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & X' \end{array} \right],$$

where $X' \in \mathfrak{o}_{n-1}$ has full rank $n-1$. Then neither I nor Y maps $\ker X$ into $\text{im } X$,

and hence the O_n -modules that they represent are not contained in $\text{RND}(\mathfrak{o}_n)$. We conclude that $\text{RND}(\mathfrak{o}_n) = \mathfrak{o}_n$, so that \mathfrak{o}_n is maximal singular.

Suppose now that G is semisimple, and choose a Borel subgroup B in G and a maximal torus T in B . Denote by \mathfrak{b} the Lie algebra of B , and by \mathfrak{h} the Lie algebra of T . Then V is the direct sum of its \mathfrak{h} -weight spaces V_λ , $\lambda \in \mathfrak{h}^*$, and we have $r = \dim V - \dim V_0$: indeed, any element of \mathfrak{h} acts by 0 on V_0 , and hence the elements of

$$\mathfrak{h}_{\text{reg}} := \{X \in \mathfrak{h} \mid \lambda(X) \neq 0 \text{ for all } \lambda \in \mathfrak{h}^* \setminus \{0\} \text{ with } V_\lambda \neq 0\}$$

have the maximal possible rank on V among elements of \mathfrak{h} , namely $\dim V - \dim V_0$. Now any semisimple element of \mathfrak{g} is tangent to a (maximal) torus [2, Section 7 or Proposition 11.8], and hence lies in a G -translate of \mathfrak{h} by the conjugacy of maximal tori. As the semisimple elements contain a dense open subset of \mathfrak{g} , so does the union of the G -translates of $\mathfrak{h}_{\text{reg}}$. Hence $\text{rk}(\rho(X)) = \dim V - \dim V_0$ for generic $X \in \mathfrak{g}$.

We would like to determine, for each \mathfrak{b} -highest weight λ of $\text{End}(V)$, the multiplicity of λ among the highest weights in $\text{RND}(\rho(\mathfrak{g}))$. It seems hard, however, to give a general formula for these multiplicities, even for irreducible modules V . Therefore, Section 5 treats an algorithm for computing them.

On the other hand, in the situation of Theorems 1 and 2 we will prove that $\text{RND}(\rho(\mathfrak{g}))$ is actually equal to $\rho(\mathfrak{g})$. For this purpose we need the following characterisation of the rank-neutral directions:

$$\text{RND}(\rho(\mathfrak{g})) = \left\{ Y \in \text{End}(V) \mid (gY)V_0 \subseteq \bigoplus_{\lambda \neq 0} V_\lambda \text{ for all } g \in G \right\}, \quad (**)$$

where we write gY for $\rho(g)Y\rho(g)^{-1}$. Indeed, an element of $\rho(\mathfrak{h}_{\text{reg}})$ has kernel V_0 and image $V_1 := \bigoplus_{\lambda \neq 0} V_\lambda$, so that the inclusion \subseteq follows from the definition of RND and the G -stability of $\text{RND}(\rho(\mathfrak{g}))$. On the other hand, if every map in the G -orbit of Y maps V_0 into V_1 , then Y is tangent to R_r at all points of $\{gX \mid g \in G, X \in \rho(\mathfrak{h}_{\text{reg}})\}$. This set is dense in $\rho(\mathfrak{g})_{\text{reg}}$, and because the set $\{X \in \rho(\mathfrak{g})_{\text{reg}} \mid Y \in T_X R_r\}$ is closed in $\rho(\mathfrak{g})_{\text{reg}}$, we conclude that $Y \in \text{RND}(\rho(\mathfrak{g}))$.

We now proceed with the proof of Theorem 1. Let G be SL_m , choose B to be the subgroup of G consisting of upper triangular matrices, and let T be the group of diagonal matrices in B . Let k be a natural number, and let $\rho : G \rightarrow \text{GL}(V)$ be the representation of G on the space V of homogeneous polynomials on K^m of degree k ; we continue to denote the corresponding representation $\mathfrak{g} \rightarrow \text{End}(V)$ also by ρ . The image $\rho(\mathfrak{g})$ is spanned by the (restrictions to V of the) differential operators $x_i \partial / \partial x_j$, $i \neq j$, and $x_i \partial / \partial x_i - x_j \partial / \partial x_j$, where the latter span $\rho(\mathfrak{h})$. The weight spaces in V are one-dimensional and spanned by the monomials $x_1^{a_1} \dots x_m^{a_m}$ with $a_1 + \dots + a_m = k$. The highest weight vector in $\rho(\mathfrak{g})$ is $x_m \partial / \partial x_1$. To apply Proposition 3 we compute the highest weight vectors in $\text{End}(V)$.

LEMMA 12. *The highest weight vectors in $\text{End}(V) = V \otimes V^*$ are precisely the powers $(x_m \partial / \partial x_1)^d$ for $d = 0, \dots, k$.*

Proof. It is clear that these are (non-zero) highest weight vectors; that there are no others follows by a dimension computation. Alternatively, the lemma is an easy application of the Pieri rule [10]. \square

The space $\rho(\mathfrak{g})$ is singular if and only if k is a multiple of m , say $k = em$, and then $x_1^e x_2^e \dots x_m^e$ spans the zero-weight space. The space $\rho(\mathfrak{sl}_m)$ has no chance of being maximal singular if $m = 2$ (unless $k = 2$), as then $\dim(V) = k + 1 = 2e + 1$ is odd and SL_2 leaves invariant a non-degenerate symmetric bilinear form on V ; $\rho(\mathfrak{g})$ is then contained in the larger singular space of linear maps that are skew relative to this bilinear form. This explains the condition ' $m \geq 3$ ' in Theorem 1.

Proof of Theorem 1. As indicated at the end of the introduction, it suffices to prove the theorem under the condition that K is algebraically closed. By Proposition 3, Lemma 12, and the characterisation $(**)$ of $\mathrm{RND}(\rho(\mathfrak{g}))$, we need to prove that if $d \in \{0, 2, 3, \dots, me\}$, then some element of the SL_m -orbit of $(x_m \partial/\partial x_1)^d$ does not map $x_1^e \dots x_m^e$ into the space spanned by all other monomials. This SL_m -orbit contains the differential operators of the form

$$\left((x_1 + \dots + x_m) \left(\alpha_1 \frac{\partial}{\partial x_1} + \dots + \alpha_m \frac{\partial}{\partial x_m} \right) \right)^d$$

with $\alpha_1 + \dots + \alpha_m = 0$. We will prove in Lemma 13 below that the coefficient of the monomial $x_1^e \dots x_m^e$ in

$$\left((x_1 + \dots + x_m) \left(\alpha_1 \frac{\partial}{\partial x_1} + \dots + \alpha_m \frac{\partial}{\partial x_m} \right) \right)^d x_1^e \dots x_m^e,$$

regarded as a polynomial in $\mathbb{Z}[\alpha_1, \dots, \alpha_m]$, is not a multiple of $\alpha_1 + \dots + \alpha_m$ for any $d \in \{0, 2, 3, \dots, me\}$. Hence, for all d in this range there exist $\alpha_1, \dots, \alpha_m \in K$ with $\alpha_1 + \dots + \alpha_m = 0$ for which that coefficient is non-zero, as required. \square

For the proof of Lemma 13 we need the following notation: \mathbb{N} denotes the set of non-negative integers, for a multi-index $\mathbf{a} \in \mathbb{N}^m$ we write $|\mathbf{a}| = a_1 + \dots + a_m$, and

$$\binom{|\mathbf{a}|}{\mathbf{a}} := \frac{|\mathbf{a}|!}{\prod_i a_i!}$$

denotes the number of ways of partitioning $\{1, \dots, |\mathbf{a}|\}$ into m labelled sets with cardinalities a_1, \dots, a_m . Also, if $\alpha = (\alpha_1, \dots, \alpha_m)$ is a list of variables, then $\alpha^{\mathbf{a}}$ abbreviates $\prod_i \alpha_i^{a_i}$. Furthermore, for $a \in \mathbb{N}$ we write $(e)_a := e(e-1) \dots (e-a+1)$ for the falling factorial, which will be used both for $e \in \mathbb{Z}$ and for e a variable, in which case it should be understood as a polynomial in $\mathbb{Q}[e]$.

LEMMA 13. *For any integers $e, m > 0$ and $d \in \{0\} \cup \{2, 3, \dots, me\}$, the coefficient $P_{d,e}$ of $x_1^e \dots x_m^e$ in*

$$\left((x_1 + \dots + x_m) \left(\alpha_1 \frac{\partial}{\partial x_1} + \dots + \alpha_m \frac{\partial}{\partial x_m} \right) \right)^d x_1^e \dots x_m^e,$$

viewed as a polynomial in $\mathbb{Z}[\alpha_1, \dots, \alpha_m]$, is not a multiple of $\alpha_1 + \dots + \alpha_m$.

Proof. Observe that the operator of multiplication with $f := x_1 + \dots + x_m$ commutes with the derivation $\delta := (\alpha_1 \partial/\partial x_1 + \dots + \alpha_m \partial/\partial x_m)$ if $\alpha_1 + \dots + \alpha_m = 0$. We therefore have two ways of computing $P_{d,e}$ modulo $\alpha_1 + \dots + \alpha_m$: we may either first apply δ^d to $x_1^e \dots x_m^e$ and then multiply the result with f^d , or the other way

around. The first computation yields

$$P_{d,e} \equiv \sum_{\mathbf{a} \in \mathbb{N}^m : |\mathbf{a}|=d} \binom{d}{\mathbf{a}}^2 \left(\prod_i (e)_{a_i} \right) \alpha^{\mathbf{a}} \pmod{\alpha_1 + \dots + \alpha_m}, \quad (***)$$

while the second computation yields

$$P_{d,e} \equiv \sum_{\mathbf{a} \in \mathbb{N}^m : |\mathbf{a}|=d} \binom{d}{\mathbf{a}}^2 \left(\prod_i (e + a_i)_{a_i} \right) \alpha^{\mathbf{a}} \pmod{\alpha_1 + \dots + \alpha_m}.$$

Now we regard e as a variable and the two expressions above as polynomials in $\mathbb{Q}[e, \alpha_1, \dots, \alpha_m]$; only later will we specialise e to appropriate values.

Substituting $-e-1$ for e in $(***)$ and using the identity $(-e-1)_a = (-1)^a (e+a)_a$ for $a \in \mathbb{N}$, we obtain the second expression for $P_{d,e}$ up to a sign; hence $P_{d,e}$ satisfies the following ‘functional equation’:

$$P_{d,-e-1} \equiv (-1)^d P_{d,e} \pmod{\alpha_1 + \dots + \alpha_m}.$$

We will now use this functional equation to prove the lemma in the case where d is small compared to e . For $d = 0$ the lemma is trivial, so we may assume that $d \geq 2$.

Let $Q_d = Q_d(e)$ denote the coefficient of $\alpha_1^{d-1} \alpha_2$ in the polynomial

$$P_{d,e}(\alpha_1, \dots, \alpha_{m-1}, -\alpha_1 - \dots - \alpha_{m-1});$$

then Q_d is a polynomial in $\mathbb{Q}[e]$ of degree at most d , whose zeroes can be determined explicitly as follows.

(i) We show that Q_d is not the zero polynomial by showing that it has a non-zero linear term. To see this, observe that the term

$$t_{\mathbf{a}} := \binom{d}{\mathbf{a}}^2 \left(\prod_i (e)_{a_i} \right) \alpha_1^{a_1} \dots \alpha_{m-1}^{a_{m-1}} (-\alpha_1 - \dots - \alpha_{m-1})^{a_m}$$

can give a non-zero contribution to Q_d only if $a_1 \leq d-1$, $a_2 \in \{0, 1\}$, and $a_3 = \dots = a_{m-1} = 0$. On the other hand, expanding $t_{\mathbf{a}}$ we encounter linear terms in e only if all a_i are zero except for one, which is then equal to d . We conclude that only the term $t_{\mathbf{a}}$ with $a_1 = \dots = a_{m-1} = 0$ and $a_m = d$ contributes to the linear term in Q_d , and a straightforward computation shows that this contribution is $-d! \neq 0$.

(ii) We claim that $Q_d = 0$ for all $e \in \mathbb{N}$ with $2e < d-1$. Indeed, if the term $t_{\mathbf{a}}$ above contributes to Q_d at all, then we have $a_1 + a_m \geq d-1$. But if $2e < d-1$, this means that at least one of the factors $(e)_{a_1}$ and $(e)_{a_m}$ is zero; hence the claim holds.

(iii) Now Q_d inherits the functional equation from $P_{d,e}$:

$$Q_d(-e-1) = (-1)^d Q_d(e),$$

which implies that the zeroes of Q_d are symmetric around $-\frac{1}{2}$. Together with the previous two observations and the fact that $\deg Q_d \leq d$, this shows that if $d = 2l$ is even, then the zeroes of Q_d are precisely

$$-l, -l+1, \dots, l-2, l-1,$$

while for $d = 2l+1$ odd, Q_d has the same zeroes plus, at most, one zero at $-\frac{1}{2}$. (In fact, one can verify that Q_{2l+1} has degree $2l+1$, so that $-\frac{1}{2}$ is indeed a zero, but we do not need that here.)

We conclude that $Q_d(e)$ is not zero for $2e \geq d-1$, and hence $P_{d,e}$ is not divisible by $\alpha_1 + \dots + \alpha_m$ for d, e in this range.

Suppose next that $e \in \mathbb{N}$ with $1 < 2e \leq d$. Then we prove that the symmetric polynomial $(***)$ is not divisible by $\alpha_1 + \dots + \alpha_m =: \sigma_1$ by rewriting it as a polynomial in the elementary symmetric polynomials

$$\sigma_l := \sum_{i_1 < i_2 < \dots < i_l} \alpha_{i_1} \dots \alpha_{i_l}, \quad l = 1, \dots, m.$$

The first step of this rewriting process goes as follows [19]: one determines the largest non-zero term $c_{\mathbf{a}} \alpha^{\mathbf{a}}$ ($\mathbf{a} \in \mathbb{N}^m$, $c_{\mathbf{a}} \in \mathbb{Z}$) in $(***)$ relative to the lexicographic order on \mathbb{N}^n defined by

$$\mathbf{a} \leq \mathbf{a}' : \Longleftrightarrow \mathbf{a} = \mathbf{a}' \text{ or the smallest } i \text{ for which } a_i \neq a'_i \text{ satisfies } a_i < a'_i.$$

Then the unique monomial $\sigma_{\mathbf{a}}$ in the σ_l having the same largest monomial $\alpha^{\mathbf{a}}$ in the α_i will appear with coefficient $c_{\mathbf{a}}$ in the end result. To carry out this step concretely, we write $d = qe + r$ with $r \in \{0, \dots, e-1\}$. Then the largest monomial in $(***)$ is

$$\alpha_1^e \dots \alpha_q^e \cdot \alpha_{q+1}^r$$

and the monomial in the σ_l having this as largest monomial is $\sigma_{q+1}^r \sigma_q^{e-r}$: the factor σ_{q+1}^r has largest monomial $\alpha_1^r \dots \alpha_{q+1}^r$ and the factor σ_q^{e-r} has largest monomial $\alpha_1^{e-r} \dots \alpha_q^{e-r}$. As by assumption $q \geq 2$, the monomial $\sigma_{q+1}^r \sigma_q^{e-r}$ is not divisible by σ_1 , and the proof is complete. \square

We now prove the rank-criticality of the adjoint images of semisimple Lie algebras.

Proof of Theorem 2. Again, it suffices to prove the theorem in the case where K is algebraically closed. We will prove $\text{RND}(\text{ad}(\mathfrak{g})) = \text{ad}(\mathfrak{g})$ and then apply Proposition 3. Therefore, let A be a rank-neutral direction of $\text{ad}(\mathfrak{g})$, and let $x \in \mathfrak{g}$ have centraliser $\ker \text{ad}(x) = \mathfrak{g}^x$ of minimal dimension; then

$$A\mathfrak{g}^x \subseteq \text{im ad}(x) = [x, \mathfrak{g}].$$

It follows from this that if $[x, y] = 0$, then the Killing form κ of \mathfrak{g} vanishes on (x, Ay) . As the commuting variety of \mathfrak{g} is irreducible [21], this implies that $\kappa(x, Ay) = 0$ for all $x, y \in \mathfrak{g}$ with $[x, y] = 0$, independent of $\dim \mathfrak{g}^x$. Hence $\text{RND}(\rho(\mathfrak{g}))$ is contained in the space

$$M(\mathfrak{g}) := \{A \in \text{End}(\mathfrak{g}) \mid \kappa(x, Ay) = 0 \text{ for all } x, y \in \mathfrak{g} \text{ with } [x, y] = 0\};$$

we claim that it is equal to $\text{ad}(\mathfrak{g})$. First, assume that this is true for *simple* \mathfrak{g} , let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be a decomposition of \mathfrak{g} into simple ideals, let $A \in M(\mathfrak{g})$, and let $y \in \mathfrak{g}_i$. Then Ay is κ -perpendicular to $\bigoplus_{j \neq i} \mathfrak{g}_j$, so $Ay \in \mathfrak{g}_i$. In other words, every \mathfrak{g}_i is A -stable, and of course $A|_{\mathfrak{g}_i} \in M(\mathfrak{g}_i)$. By assumption there exist $z_i \in \mathfrak{g}_i$ such that $A|_{\mathfrak{g}_i} = \text{ad}_{\mathfrak{g}_i} z_i$, and then $A = \sum_i \text{ad}_{\mathfrak{g}}(z_i)$.

It remains to prove that $M(\mathfrak{g}) = \text{ad}(\mathfrak{g})$ for simple \mathfrak{g} . For \mathfrak{sl}_2 this is easy, so we may suppose that \mathfrak{g} has rank at least 2. Setting $x = y$ in the condition on A , we see that $M(\mathfrak{g}) \subseteq \mathfrak{o}(\kappa)$, the orthogonal Lie algebra defined by κ . Moreover, $M(\mathfrak{g})$ is stable under conjugation with any automorphism of \mathfrak{g} , and this implies two things: first, that $M(\mathfrak{g})$ is a \mathfrak{g} -module and second, using the Chevalley involution of \mathfrak{g} , that it is self-dual as a \mathfrak{g} -module. Now the \mathfrak{g} -module $\mathfrak{o}(\kappa)/\text{ad}(\mathfrak{g})$ is irreducible if \mathfrak{g} is not of type A – its highest weight can be determined explicitly but is not of

interest to us – while it is a direct sum $W \oplus W^*$ for some non-self-dual module W if \mathfrak{g} is of type A . In any case, $M(\mathfrak{g})$ is either $\text{ad}(\mathfrak{g})$ or $\mathfrak{o}(\kappa)$. But $M(\mathfrak{g}) \neq \mathfrak{o}(\kappa)$: choose for instance x, y in a Cartan subalgebra of \mathfrak{g} (so that $[x, y] = 0$) satisfying $\kappa(x, x) = \kappa(y, y) = 1 - \kappa(x, y) = 1$, and let A be the map sending x to y , y to $-x$, and the κ -orthogonal complement of $\langle x, y \rangle_K$ to 0; then $A \in \mathfrak{o}(\kappa) \setminus M(\mathfrak{g})$. We conclude that

$$M(\mathfrak{g}) = \text{RND}(\text{ad}(\mathfrak{g})) = \text{ad}(\mathfrak{g}),$$

as claimed. \square

5. Some computer results

Judging by the difficulty that we had in proving Theorems 1 and 2, it seems hard to find a general formula for the highest weights in $\text{RND}(\rho(\mathfrak{g}))$, given a representation $\rho : G \rightarrow \text{GL}(V)$ of a semisimple algebraic group G over an algebraically closed field of characteristic zero. In this section we indicate an algorithm for computing these highest weights in concrete situations, and present some results obtained with this algorithm. First, we retain the setting and notation of Section 3. A randomised algorithm to compute the tangent space $T_{N_{\text{reg}}}Z$ is based on the following observation.

LEMMA 14. *For all non-negative integers l and e , the set of $(n_1, \dots, n_l) \in N_{\text{reg}}^l$ for which*

$$\dim \bigcap_{i=1}^l T_{n_i}Z \geq e$$

is a closed subvariety of N_{reg}^l .

Proof. Let d be the dimension of Z , and let $\gamma : Z_{\text{reg}} \rightarrow \text{Gr}_d(M)$ be the Gauss map sending a point of Z_{reg} to its tangent space, regarded as a point in the Grassmannian $\text{Gr}_d(M)$ of d -dimensional subspaces of M . Now the set of all l -tuples $(T_1, \dots, T_l) \in \text{Gr}_d(M)^l$ whose intersection has dimension at least e is closed in $\text{Gr}_d(M)^l$, and hence so is its pre-image under $(\gamma|_{N_{\text{reg}}})^{\times l}$, which is precisely the set of the lemma. \square

For dimension reasons, the space $T_{N_{\text{reg}}}Z$ is the intersection of *finitely many* tangent spaces $T_{n_i}Z$, $i = 1, \dots, m$, with $n_i \in N_{\text{reg}}$. The preceding lemma suggests the following randomised algorithm to compute $T_{N_{\text{reg}}}Z$. First, find an upper bound on m , and second, choose m elements of N at random. These are probably smooth points of Z by (*), and by the preceding proposition the intersection of their tangent spaces is probably equal to $T_{N_{\text{reg}}}Z$. In particular, if this intersection is equal to N , then we are *sure* that N is a maximal vector space in Z .

We now outline a version of this algorithm for the computation of the highest weights in the G -module $\text{RND}(\rho(\mathfrak{g}))$, where $\rho : G \rightarrow \text{GL}(V)$ is a representation of a semisimple algebraic group as in Section 4.

(1) Compute the non-zero highest weight spaces HW_{μ} of $\text{End}(V)$ relative to a Borel subalgebra \mathfrak{b} of \mathfrak{g} and a Cartan subalgebra \mathfrak{h} contained in \mathfrak{b} ; this is elementary linear algebra.

(2) For each of these highest weight spaces, say HW_{μ} of dimension l , choose l random elements $X_1, \dots, X_l \in \rho(\mathfrak{g})$, verify that they have maximal rank, and

compute the subspace

$$\{Y \in \mathrm{HW}_\mu \mid Y \ker X_i \subseteq \mathrm{im} X_i \text{ for all } i = 1, \dots, l\}.$$

(3) The dimension of this space is the multiplicity of μ among the highest weights in $\mathrm{RND}(\rho(\mathfrak{g}))$. More precisely, it is an upper bound for this multiplicity, which with high probability is sharp.

If we find that the multiplicity of every highest weight in $\mathrm{RND}(\rho(\mathfrak{g}))$ is equal to its multiplicity in $\rho(\mathfrak{g})$, then $\rho(\mathfrak{g})$ is rank-critical, by Proposition 3. In my implementation of this algorithm in GAP [12], I worked with the split form $\mathfrak{g}_\mathbb{Q}$ over \mathbb{Q} of the semisimple Lie algebra \mathfrak{g} ; in particular, the X_i used in the algorithm were taken from $\mathfrak{g}_\mathbb{Q}$. *A priori*, the \mathfrak{g} -module thus found may be larger than $\mathrm{RND}(\rho(\mathfrak{g}))$, as the intersection of the tangent spaces is taken only over rational X_i . However, from the facts that $\mathfrak{g}_\mathbb{Q}$ is (Zariski) dense in \mathfrak{g} , and that for given $Y \in \mathrm{End}(V)$ the set of $X \in \rho(\mathfrak{g})_{\mathrm{reg}}$ with $Y \in T_X R_r$ is closed in \mathfrak{g} , it follows readily that the module obtained using $\mathfrak{g}_\mathbb{Q}$ rather than \mathfrak{g} is, indeed, $\mathrm{RND}(\rho(\mathfrak{g}))$. I list some examples found by experiments with this algorithm.

(1) The images of the adjoint representations of simple Lie algebras of types $A_1, \dots, A_4, B_2, \dots, B_4, C_3, C_4$, and G_2 were proved to be rank-critical with this algorithm. This computational evidence led to the formulation of Theorem 2.

(2) The image of the 26-dimensional representation of F_4 is rank-critical (of rank 24).

(3) Let \mathfrak{g} be simple of type G_2 , and let ρ be the 7-dimensional representation of highest weight $[1, 0]$ (relative to the numbering of the simple roots as in [3]), with zero-weight multiplicity 1. Then $\mathrm{RND}(\rho(\mathfrak{g}))$ is equal to \mathfrak{o}_7 , which of course is still singular, so $\rho(\mathfrak{g})$ is not a maximal singular space.

(4) Similarly, if \mathfrak{g} is of type G_2 and ρ is the 27-dimensional representation of highest weight $[2, 0]$ with zero-weight multiplicity 3, then $\mathrm{RND}(\rho(\mathfrak{g}))$ is equal to $\phi(\mathfrak{o}_7)$, where ϕ is the representation of \mathfrak{o}_7 of highest weight $[2, 0, 0]$, which restricts to ρ on \mathfrak{g} . As both $\rho(\mathfrak{g})$ and $\phi(\mathfrak{o}_7)$ have generic rank 24, the former is not rank-critical, but the latter is.

(5) Let ρ be the 35-dimensional irreducible representation of $\mathfrak{g} = \mathfrak{sl}_3$ of highest weight $[4, 1]$. Then $\mathrm{RND}(\rho(\mathfrak{g}))$ is a sum of three irreducible modules of highest weights $[1, 4]$, $[1, 1]$, and $[4, 1]$. Hence, Proposition 3 cannot be applied to conclude rank-criticality. Note that by the results of Dynkin [6] the image of ρ is a maximal subalgebra of \mathfrak{sl}_{35} , so that there is no easy argument, as in the previous two examples, which shows that $\rho(\mathfrak{g})$ is *not* rank-critical.

6. Conclusion and further questions

Representations of semisimple Lie algebras yield an abundance of rank-critical matrix spaces, which suggests that the (commonly believed to be intractable) classification of such spaces may in some sense include the classification of Lie algebra representations. In particular, among the representations of \mathfrak{sl}_3 we found infinitely many where the image of \mathfrak{sl}_3 is a maximal singular space. The matrix spaces \mathcal{A} constructed this way actually satisfy an *a priori* stronger condition than rank-criticality, namely: $\mathcal{A} = \mathrm{RND}(\mathcal{A})$. These results pose many questions for further research, of which the following seem the most interesting.

(1) Directly describe, given the highest weights in a representation ρ of a semi-simple Lie algebra \mathfrak{g} , the highest weights in the \mathfrak{g} -module $\text{RND}(\rho(\mathfrak{g}))$. The present proofs of Theorems 1 and 2 are somewhat *ad hoc*, and the algorithm of Section 5 is computationally rather intensive, so that it works only for representations of dimensions at most 50 or so.

(2) Investigate the discrepancy between rank-criticality and the condition $\mathcal{A} = \text{RND}(\mathcal{A})$.

(3) For m_n the minimal dimension of a maximal singular space of $n \times n$ -matrices, determine $\liminf_{n \rightarrow \infty} m_n$; this number is larger than 2 and at most 8, as we have seen, while – to the best of my knowledge – it was previously believed to be infinite. Also, determine $\limsup_{n \rightarrow \infty} m_n$.

(4) Investigate whether the maximal singular spaces of Theorem 1 remain maximal modulo primes, and, more generally, whether rank-critical spaces constructed with the algorithm of Section 5 remain rank-critical modulo primes.

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